Examples of backreaction of small scale inhomogeneities in cosmology

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Abstract

In previous work, we introduced a new framework to treat large scale backreaction effects due to small scale inhomogeneities in general relativity. We considered one-parameter families of spacetimes for which such backreaction effects can occur, and we proved that, provided the weak energy condition on matter is satisfied, the leading effect of small scale inhomogeneities on large scale dynamics is to produce a traceless effective stress-energy tensor that itself satisfies the weak energy condition. In this work, we illustrate the nature of our framework by providing two explicit examples of one-parameter families with backreaction. The first, based on previous work of Berger, is a family of polarized vacuum Gowdy spacetimes on a torus, which satisfies all of the assumptions of our framework. As the parameter approaches its limiting value, the metric uniformly approaches a smooth background metric, but spacetime derivatives of the deviation of the metric from the background metric do not converge uniformly to zero. The limiting metric has nontrivial backreaction from the small scale inhomogeneities, with an effective stress-energy that is traceless and satisfies the weak energy condition, in accord with our theorems. Our second one-parameter family consists of metrics which have a uniform Friedmann-Lemaître-Robertson-Walker limit. This family satisfies all of our assumptions with the exception of the weak energy condition for matter. In this case, the limiting metric has an effective stress-energy tensor which is not traceless. We emphasize the importance of imposing energy conditions on matter in studies of backreaction.

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I. INTRODUCTION

The standard model of cosmology is based on a hierarchy of length scales. At the largest scales in cosmology, it is generally believed that the universe is well-described by a metric with Friedmann-Lemaître-Robertson-Walker (FLRW) symmetry that satisfies Einstein's equation with source given by the averaged stress-energy of matter. At scales somewhat smaller than this, the deviations from the FLRW solution are believed to be well described by linear perturbation theory. Finally, below a certain scale, nonlinear effects become important, and numerical N-body simulations are normally used to make predictions. The above beliefs underlie the standard cosmological model, which has proven to be extremely successful to date.

Despite the success of the standard cosmological model, objections have been raised that nonlinear effects due to the small scale structure are not being fully taken into account (see, e.g., [1]). The Friedmann equations neglect averages of terms nonlinear in small scale perturbations, which are present in the Einstein equation. Since nonlinearities are important for small scale dynamics, how do we know that their averaged effects cannot contribute at large scales? Might it even be possible that such nonlinearities could produce effects on large scale dynamics that mimic the effect of a cosmological constant?

At small scales, where nonlinear dynamics are important, fractional density perturbations can be much larger than 1. This formed the basis of Ellis's original argument [1] that neglecting nonlinear terms in the Einstein equation could be problematic. However, Ishibashi and Wald [2] argued that, even though perturbations in the matter stress-energy can be large, perturbations in the metric (as opposed to derivatives of the metric) should nevertheless be small, provided that appropriate gauge choices are made. They argued that, except in the immediate vicinity of strong field objects such as black holes and neutron stars, our universe should be well described on all scales by a Newtonianly perturbed FLRW metric, and that the terms in Einstein's equation that are nonlinear in the deviation of the metric from an FLRW metric should be negligible.

However, it is clear that, in general, non-Newtonian circumstances, even when the deviation of the metric from a metric with FLRW symmetry is small, the nonlinear terms in the deviation of the metric from FLRW symmetry *can* affect the large scale dynamics. Indeed, if the universe were filled with low amplitude, high frequency gravitational radiation, then the

metric could be uniformly close to a metric with FLRW symmetry, but this FLRW metric would behave dynamically as though the universe were filled with a $P = \frac{1}{3}\rho$ fluid. This dynamical behavior occurs as a consequence of the nonlinear terms in Einstein's equation. Thus, an interesting issue in general relativity is to quantify the degree to which small scale spacetime inhomogeneities can, through nonlinear interactions, affect large scale dynamics.

Prior to our work [3], a number of averaging formalisms had been developed to try to address this issue. The most widely studied approach was initiated by Buchert [4, 5]. Here, one works in comoving, synchronous coordinates, so that the metric takes the form

$$ds^2 = -dt^2 + q_{ij}(t, x)dx^i dx^j. \tag{1.1}$$

One can then define averages of scalars over (compact, comoving regions of) constant-t hypersurfaces. In particular, Buchert defines an averaged "scale factor" as the cube root of the volume of one of these regions. Equations of motion for the averaged quantities are obtained by averaging (components of) Einstein's equation. In particular, by averaging the Hamiltonian constraint and the Raychaudhuri equation, analogs of the Friedmann equations can be derived in this framework.

There are two independent serious difficulties with this framework. First, it is far from clear how to physically interpret the averaged quantities defined in this framework. For example, if one were to set up synchronous coordinates of the form (1.1) in our solar system based upon a family of geodesics that are initially (nearly) "at rest" with respect to the sun, then the geodesics near the planets would immediately begin shearing; the nonlinear effects of shear would soon result in significant convergence and eventually there would be caustics. The dynamics of the "scale factor" (i.e., the volume of a co-moving region) would be quite complicated and would show large effects produced by the planets. One might then be tempted to conclude that nonlinear effects produced by the planets result in large effects on the spacetime metric of the solar system. Obviously, this is not the case: The "large effects" are gauge artifacts resulting from the use of synchronous coordinates rather than coordinates in which the metric is (nearly) stationary, and quantities like the "scale factor" do not have any physical interpretation in terms of the observations of the metric made by (nearly) stationary observers.

A second serious difficulty of this approach is that the "Friedmann equations" derived in this framework are not closed. They contain a scalar quantity $\mathcal{Q}_{\mathcal{D}}$, arising from averaging

nonlinear terms in the Einstein equation, which is called the *kinematical backreaction* scalar. The evolution of $\mathcal{Q}_{\mathcal{D}}$ is undetermined in the framework. This is because the framework considers only the spatial average of two components of Einstein's equation, and thus is discarding much of the content of Einstein's equation.

Another approach to backreaction, known as *macroscopic gravity*, was initiated by Zalaletdinov [6]. Rather than restricting to averaging scalar quantities on spatial slices, in macroscopic gravity, an averaging operator $\langle\cdot\rangle$ is introduced which allows one to define local spacetime averages of tensor quantities. The idea is to derive an equation for the average $\langle g_{ab}\rangle$ of the actual spacetime metric g_{ab}. Backreaction in this framework occurs through the "connection correlation tensor",

$$Z^{a}{}_{bc}{}^{d}{}_{ef} = \langle \Gamma^{a}{}_{b[c}\Gamma^{d}{}_{|e|f]}\rangle - \langle \Gamma^{a}{}_{b[c}\rangle\langle \Gamma^{d}{}_{|e|f]}\rangle, \tag{1.2}$$

where $\Gamma^{a}{}_{bc}$ is the Christoffel connection. Various properties are derived for Z_{abcdef}, but, as with $\mathcal{Q}_{\mathcal{D}}$, the full content of Einstein's equation is not used.

Because of the presence of undetermined quantities in the averaged frameworks of Buchert and Zalaletdinov, a common approach to "solve" the equations has been to *assume* a particular form for the backreaction tensors, and from that starting point, derive results for the large scale behavior [7, 8]. However, no argument is given that the assumed form of the backreaction tensors can actually arise from averaging a physically reasonable small scale matter distribution. For example, in [8], Buchert and Obadia assume that $\mathcal{Q}_{\mathcal{D}}$ is equivalent to a scalar field (called the *morphon field* [9]), with a particular potential. With a suitable choice of potential, they find that inflation can occur in vacuum spacetimes due to the presence of inhomogeneities. However, one is not free to arbitrarily specify the large scale effects of backreaction: They must be shown to arise from averaging an actual inhomogeneous spacetime.

Our approach—which we summarize in detail in section Sec. II—bears considerable resemblance to that of Zalaletdinov [6] in that we define a backreaction tensor μ_{abcdef}, which is closely related to Zalaletdinov's Z_{abcdef}. However, we make full use of Einstein's equation, and we have better mathematical control over the perturbations through our introduction of one-parameter families, which allow us to rigorously take a short wavelength limit. "Averaging" of nonlinear terms in Einstein's equation may then be rigorously defined via the use of weak limits. We were thus able to derive strong constraints on μ_{abcdef}, and we proved the fol-

lowing theorem about dynamical behavior in the limit of short wavelength perturbations: If the matter stress-energy tensor satisfies the weak energy condition (i.e., has positive energy density, as measured by any timelike observer), then the leading order effect of small scale inhomogeneities on the background metric is to produce an effective stress-energy tensor that is traceless and itself satisfies the weak energy condition. In particular, backreaction produced by small scale inhomogeneities cannot mimic a cosmological constant.

However, our work merely assumed one-parameter families with the properties stated in Sec. II below; we did not prove existence of a family with nontrivial backreaction, thus leaving open the possibility that our assumptions are self-consistent only in the case of no backreaction. Objections also have been raised that the interpretation of the parameter appearing in our one-parameter families is unclear [10], and that our averaging scheme is "ultralocal", and thus cannot capture effects due to matter inhomogeneities over finite regions [11]. Therefore, it seems clear that it would be useful for us to provide a concrete example of a one-parameter family of spacetimes with nontrivial backreaction that satisfies our assumptions. Such an example would serve the multiple purposes of proving the self-consistency of our assumptions, illustrating the meaning of the parameter appearing in our one-parameter families, and showing explicitly how our theorems have direct implications for perturbations of finite amplitude and wavelength.

The purpose of this paper is to provide two examples of one-parameter families of spacetimes with nontrivial backreaction. The first, analyzed in Sec. III, is adopted from [12] and is a family of vacuum Gowdy spacetimes. This family satisfies all of the assumptions of our work and has nontrivial backreaction in that the limiting metric does not satisfy the vacuum Einstein equation. Its effective stress-energy tensor can be explicitly seen to be traceless and satisfy the weak energy condition, as guaranteed by the theorems in [3].

The second family, which we produce in Sec. IV, is obtained by "Synge's method," i.e., by simply writing down a family of metrics and declaring the matter stress-energy of each spacetime to be equal to the Einstein tensor of the metric, thereby trivially "solving" Einstein's equation [13]. This family is non-vacuum and approaches a FLRW spacetime as the parameter approaches its limiting value. It satisfies all of our assumptions except that the matter stress-energy does not satisfy the weak energy condition. We obtain a nontrivial effective stress-energy tensor, which does not satisfy the properties of our theorems. This example shows that if matter violates the weak energy condition (or if Einstein's equation is

not sufficiently utilized), then it is easy to produce spacetimes with significant backreaction that is unconstrained by our theorems. This illustrates the necessity of applying the full content of Einstein's equation with physically reasonable energy conditions imposed upon matter in any analysis of backreaction.

In this work, we follow all notation and sign conventions of [14].

II. FRAMEWORK

In this section we review our framework and we summarize the main results of [3]. Our framework is a generalization to the non-vacuum case of a framework proposed by Burnett [15], which itself is a rigorous version of Isaacson's approach [16].

As indicated in the Introduction, to analyze a spacetime with backreaction $g_{ab}(\lambda_0)$, we attach to it a one-paramater family of spacetimes $g_{ab}(\lambda)$ for $0 \leq \lambda \leq \lambda_0$. We require $g_{ab}(\lambda)$ to be a solution to Einstein's equation for all $\lambda > 0$, and, as $\lambda \to 0$, we require uniform convergence of $g_{ab}(\lambda)$ to a "background metric" $g^{(0)}_{ab} \equiv g_{ab}(0)$. One-parameter families of metrics are also employed in this manner to properly treat ordinary perturbation theory (see, e.g., Sec. 7.5 of [14]). However, whereas in ordinary perturbation theory the metric is required to be jointly smooth in λ and the coordinates x, here we require only that spacetime derivatives of $(g_{ab}(\lambda) - g^{(0)}_{ab})$ are *bounded* as $\lambda \to 0$. This elevates the importance of derivatives of perturbations of the background metric, so that *a priori*, small scale inhomogeneities can play a dynamical role in the evolution of the background metric. Indeed, in contrast to ordinary perturbation theory, it is not true, in general, that the Einstein tensor has a uniform limit as $\lambda \to 0$, and it does not follow that $g^{(0)}_{ab}$ is a solution to the Einstein equation. However, if we add a suitable assumption that "spacetime averages exist", then we can derive an equation for $g^{(0)}_{ab}$, which contains the contributions arising from the backreaction produced by small scale inhomogeneities.

Our precise assumptions are as follows. Fix a spacetime manifold M with derivative operator ∇_a, and let $g_{ab}(\lambda)$ be a one-parameter family of metrics on M, defined for $\lambda \geq 0$. Let e_{ab} be an arbitrary Riemannian metric on M, and define $|t_{a_1 \dots a_n}|^2 = e^{a_1 b_1} \dots e^{a_n b_n} t_{a_1 \dots a_n} t_{b_1 \dots b_n}$. Suppose now that the following assumptions hold:

(i) For all $\lambda > 0$, we have

$$G_{ab}(g(\lambda)) + \Lambda g_{ab}(\lambda) = 8\pi T_{ab}(\lambda), \tag{2.1}$$

where $T_{ab}(\lambda)$ satisfies the weak energy condition, i.e., for all $\lambda > 0$ we have $T_{ab}(\lambda)t^a(\lambda)t^b(\lambda) \geq 0$ for all vectors $t^a(\lambda)$ that are timelike with respect to $g_{ab}(\lambda)$.

(ii) There exists a smooth positive function $C_1(x)$ on M such that

$$|h_{ab}(\lambda, x)| \leq \lambda C_1(x), \tag{2.2}$$

where $h_{ab}(\lambda, x) \equiv g_{ab}(\lambda, x) - g_{ab}(0, x)$.

(iii) There exists a smooth positive function $C_2(x)$ on M such that

$$|\nabla_c h_{ab}(\lambda, x)| \leq C_2(x). \tag{2.3}$$

(iv) There exists a smooth tensor field μ_{abcdef} on M such that

$$\operatorname*{w-lim}_{\lambda\to 0} [\nabla_a h_{cd}(\lambda)\nabla_b h_{ef}(\lambda)] = \mu_{abcdef}, \tag{2.4}$$

where "w-lim" denotes the weak limit.

The notion of "weak limit", which appears in assumption (iv), corresponds roughly to taking a local spacetime average, and then taking the limit as $\lambda \to 0$. More precisely, $A_{a_1\ldots a_n}(\lambda)$ converges *weakly* to $A^{(0)}_{a_1\ldots a_n}$ as $\lambda \to 0$ if and only if, for all smooth tensor fields $f^{a_1\ldots a_n}$ of compact support,

$$\lim_{\lambda\to 0} \int f^{a_1\ldots a_n} A_{a_1\ldots a_n}(\lambda) = \int f^{a_1\ldots a_n} A^{(0)}_{a_1\ldots a_n}. \tag{2.5}$$

Assumptions (i)–(iv) allow us to derive an equation satisfied[1] by $g^{(0)}_{ab}$. Indeed, in [3] we showed that $g_{ab}(0)$ satisfies the equation

$$G_{ab}(g^{(0)}) + \Lambda g^{(0)}_{ab} = 8\pi T^{(0)}_{ab} + 8\pi t^{(0)}_{ab}. \tag{2.6}$$

Here, $T^{(0)}_{ab} \equiv \text{w-lim}_{\lambda\to 0} T_{ab}(\lambda)$—which must exist as a result of the assumptions—and $t^{(0)}_{ab}$ is a particular linear combination of contractions of the tensor μ_{abcdef}. The tensor $t^{(0)}_{ab}$ is called the "effective gravitational stress-energy tensor," and it describes the dominant[2] contribution to backreaction due to small scale inhomogeneities.

In [3], we proved two theorems constraining $t^{(0)}_{ab}$:

[1] Note that assumption (i) only requires that the Einstein equation hold for $\lambda > 0$, but not for $\lambda = 0$.

[2] Higher order contributions to back reaction were also derived in [3]. In [17], we showed that the dominant effect of the higher order contributions on the dynamics of our universe is to modify the Friedmann equations by inclusion of effective stress-energy contributions associated with Newtonian gravitational potential energy and stresses and with the kinetic motion of matter.

Theorem 1 *Given a one-parameter family $g_{ab}(\lambda)$ satisfying assumptions (i)–(iv) above, the effective stress-energy tensor $t^{(0)}_{ab}$ appearing in equation* (2.6) *for the background metric $g^{(0)}_{ab}$ is traceless,*

$$t^{(0)a}{}_{a} = 0. \tag{2.7}$$

Theorem 2 *Given a one-parameter family $g_{ab}(\lambda)$ satisfying assumptions (i)–(iv) above, the effective stress-energy tensor $t^{(0)}_{ab}$ appearing in equation* (2.6) *for the background metric $g^{(0)}_{ab}$ satisfies the weak energy condition, i.e.,*

$$t^{(0)}_{ab} t^a t^b \geq 0 \tag{2.8}$$

for all t^a that are timelike with respect to $g^{(0)}_{ab}$.

In essence, these theorems show that only those small scale metric inhomogeneities corresponding to gravitational radiation can have a significant backreaction effect[3]. In particular, in the case where we have FLRW symmetry, the effective stress-energy tensor must be of the form of a $P = \frac{1}{3}\rho$ fluid, and therefore cannot mimic dark energy.

Finally, we note that in the statement of assumptions (i)–(iv) above, λ is a "continuous parameter," i.e., it takes all real values in an interval $[0, \lambda_0]$ for some $\lambda_0 > 0$. However, there would be no essential change if we took λ to be a "discrete parameter" in our assumptions—such as $\lambda = \{1/N\}$ for all positive integers N—provided only that λ takes positive values that limit to 0. It will be convenient to use such a discrete parameter in the example of the next section.

III. BACKREACTION IN VACUUM GOWDY SPACETIMES

Gowdy spacetimes are exact plane-symmetric vacuum solutions which describe closed cosmologies containing gravitational waves [19]. We restrict to the case where spatial slices have topology T^3. The general T^3 Gowdy metric may be written in the form [20]

$$ds^2_{\text{Gowdy}} = e^{(\tau-\alpha)/2}\left(-e^{-2\tau}d\tau^2 + d\theta^2\right) + e^{-\tau}\left[e^P d\sigma^2 + 2e^P Q d\sigma d\delta + \left(e^P Q^2 + e^{-P}\right) d\delta^2\right], \tag{3.1}$$

[3] Although matter inhomogeneities can produce only a negligibly small effect on the *dynamical evolution* of the background FLRW metric, they can produce large observable effects on the *apparent luminosity* of objects in the universe via gravitational lensing; see, e.g., [18].

where the functions α, P and Q depend only on τ and θ and the spatial coordinates have the range $0 \leq \theta, \sigma, \delta < 2\pi$, with periodic boundary conditions. We will restrict consideration to the case $Q = 0$, known as the *polarized* Gowdy spacetimes.

When $Q = 0$, the vacuum Einstein equations reduce to the evolution equation,

$$0 = \ddot{P} - e^{-2\tau} P''. \tag{3.2}$$

and the constraints,

$$\dot{\alpha} = \dot{P}^2 + e^{-2\tau} (P')^2 \,, \tag{3.3}$$

$$\alpha' = 2P'\dot{P} \,. \tag{3.4}$$

Here, the prime and dot denote differentiation with respect to θ and τ, respectively. The general solution to (3.2) is

$$P = A_0 + B_0 \tau + \sum_{n=1}^{\infty} \left[A_n J_0 \left(n e^{-\tau} \right) + B_n Y_0 \left(n e^{-\tau} \right) \right] \sin \left(n\theta + \phi_n \right) , \tag{3.5}$$

where A_n, B_n, and ϕ_n are free parameters. The discrete index n arises from the compactness of the θ direction.

Following [12] (see also [21]), we now construct a one-parameter family satisfying assumptions (i)–(iv). As discussed in the last paragraph of the previous section, our family will be parametrized by a discrete parameter N, and the limit $N \to \infty$ will correspond to $\lambda \to 0$. We choose

$$P_N = \frac{A}{\sqrt{N}} J_0 \left(N e^{-\tau} \right) \sin(N\theta), \tag{3.6}$$

where A is an arbitrary constant. The constraint equations may be solved by direct integration to yield

$$\alpha_N = -\frac{A^2 e^{-\tau}}{2} J_1 \left(N e^{-\tau} \right) J_0 \left(N e^{-\tau} \right) \cos \left(2N\theta \right) \tag{3.7}$$

$$-\frac{A^2 N e^{-2\tau}}{4} \left\{ \left[J_0 \left(N e^{-\tau} \right) \right]^2 + 2 \left[J_1 \left(N e^{-\tau} \right) \right]^2 - J_0 \left(N e^{-\tau} \right) J_2 \left(N e^{-\tau} \right) \right\} . \tag{3.8}$$

From the asymptotic form of the Bessel function, we find that for large values of N,

$$P_N \to \frac{A}{N} \sqrt{\frac{2e^{\tau}}{\pi}} \cos \left(N e^{-\tau} - \frac{\pi}{4} \right) \sin \left(N\theta \right) . \tag{3.9}$$

From this and the similar asymptotic form of α_N, it may be verified that assumptions (i)–(iv) of the previous section hold for the family $\{P_N, \alpha_N\}$. In particular, as $N \to \infty$, we have

$$\lim_{N\to\infty} P_N = 0, \tag{3.10}$$

$$\lim_{N\to\infty} \alpha_N = -\frac{A^2 e^{-\tau}}{\pi} \tag{3.11}$$

where the convergence is uniform on compact sets. Thus, the "background metric" is

$$ds^2_{(0)} = e^{\left(\tau + A^2 e^{-\tau}/\pi\right)/2}\left(-e^{-2\tau} d\tau^2 + d\theta^2\right) + e^{-\tau}\left(d\sigma^2 + d\delta^2\right). \tag{3.12}$$

Despite the fact that for each $N < \infty$, the metric is a vacuum solution, the limiting metric (3.12) is *not* a vacuum solution. The effective stress-energy, $t^{(0)}_{ab}$, could be computed by computing μ_{abcdef} via (2.4) and using the formula for $t^{(0)}_{ab}$ in terms of μ_{abcdef} given in [3]. However, it is far easier to obtain $t^{(0)}_{ab}$ by simply computing the left side of (2.6). It is easily seen that the nonvanishing components of the Einstein tensor of the metric (3.12) are given by

$$G_{\tau\tau}\left(g^{(0)}\right) = \frac{A^2 e^{-\tau}}{4\pi}$$

$$G_{\theta\theta}\left(g^{(0)}\right) = \frac{A^2 e^{\tau}}{4\pi}. \tag{3.13}$$

Note that the effective stress-energy $t^{(0)}_{ab} = G_{ab}/8\pi$ associated with (3.13) is traceless and satisfies the weak energy condition, in accord with our general theorems.

Thus, we have provided an explicit example of a one-parameter family wherein the small scale inhomogeneities produce a nontrivial backreaction. One can see explicitly in this example how, in our framework, the vacuum, inhomogeneous spacetime metric $g_{ab}(N)$ can be well approximated for large but finite N by the non-vacuum, homogeneous spacetime metric (3.12).

IV. BACKREACTION WITHOUT ENERGY CONDITIONS

In this section, we provide an example of a one-parameter family that satisfies assumptions (ii)–(iv) of Sec. II but does not satisfy the requirement of assumption (i) that the matter stress-energy tensor satisfy the weak energy condition. This illustrates the importance of the weak energy condition on matter for the validity of theorems 1 and 2 of Sec. II.

For our one-parameter family, $g_{ab}(\lambda)$, we choose metrics conformally related to a spatially flat FLRW metric $g^{(0)}_{ab}$, i.e.,

$$g_{ab}(\lambda) = \Omega^2(\lambda) g^{(0)}_{ab} = \Omega^2(\lambda) a^2(\tau) \eta_{ab}, \tag{4.1}$$

where $a(\tau)$ is chosen arbitrarily. We choose the conformal factor to be

$$\log \Omega(\lambda) = \lambda A \left[\sin\left(\frac{x}{\lambda}\right) + \sin\left(\frac{y}{\lambda}\right) + \sin\left(\frac{z}{\lambda}\right) \right], \tag{4.2}$$

with A constant. Of course, $g_{ab}(\lambda)$ does not satisfy Einstein's equation with any known form of matter. Nevertheless, we may simply declare the existence of a new form of matter with stress-energy tensor given by

$$T_{ab}(\lambda) = \frac{1}{8\pi} G_{ab}(g(\lambda)) \tag{4.3}$$

for all $\lambda > 0$. Then $g_{ab}(\lambda)$ is a solution of Einstein's equation (with vanishing cosmological constant, $\Lambda = 0$) for all $\lambda > 0$. This procedure for "solving" Einstein's equation is usually referred to as "Synge's method" [13]. It can then be easily verified that our one-parameter family (4.1) and (4.2) satisfies all of the assumptions of Sec. II except that the stress-energy tensor of our new form of matter does not satisfy the weak energy condition for $\lambda > 0$.

By (2.6), the effective stress-energy tensor produced by the small scale inhomogeneities is

$$\begin{aligned} t^{(0)}_{ab} &= \frac{1}{8\pi} G_{ab}(g^{(0)}) - T^{(0)}_{ab} \\ &= \frac{1}{8\pi} \operatorname*{w-lim}_{\lambda \to 0} \left[G_{ab}(g^{(0)}) - G_{ab}(g(\lambda)) \right] \\ &= \frac{1}{8\pi} \operatorname*{w-lim}_{\lambda \to 0} \Big[2 \nabla_a \nabla_b \log \Omega - 2 g^{(0)}_{ab} g^{(0)cd} \nabla_c \nabla_d \log \Omega \\ &\quad -2 \left(\nabla_a \log \Omega\right) \left(\nabla_b \log \Omega\right) - g^{(0)}_{ab} g^{(0)cd} \left(\nabla_c \log \Omega\right) \left(\nabla_d \log \Omega\right) \Big] . \end{aligned} \tag{4.4}$$

Here, on the last line, we used the expression relating the two Ricci tensors (see appendix D of [14]),

$$\begin{aligned} R_{ab}(g(\lambda)) = R_{ab}(g^{(0)}) &- 2 \nabla_a \nabla_b \log \Omega - g^{(0)}_{ab} g^{(0)cd} \nabla_c \nabla_d \log \Omega \\ &+ 2 \left(\nabla_a \log \Omega\right) \left(\nabla_b \log \Omega\right) - 2 g^{(0)}_{ab} g^{(0)cd} \left(\nabla_c \log \Omega\right) \left(\nabla_d \log \Omega\right), \end{aligned} \tag{4.5}$$

where ∇_a is the derivative operator associated with $g^{(0)}_{ab}$. A straightforward calculation[4]

[4] The only nontrivial weak limits involved are of the form $\operatorname{w-lim}_{\lambda \to 0} \cos^2(x/\lambda) = 1/2$.

yields

$$t_{00}^{(0)} = \frac{3}{16\pi}A^2 \tag{4.6}$$

$$t_{0i}^{(0)} = 0 \tag{4.7}$$

$$t_{ij}^{(0)} = -\frac{5}{16\pi}A^2\delta_{ij}\,. \tag{4.8}$$

Thus, our effective stress-energy tensor produced by small-scale inhomogeneities corresponds to a $P = -\frac{5}{3}\rho$ fluid. This fails[5] to be traceless and fails to satisfy the weak energy condition, in violation of both theorems 1 and 2 of Sec. II. This should come as no surprise, since we have failed to satisfy assumption (i). This example thus shows that assumption (i) is essentially needed for the validity of our main results.

We note that several analyses [7, 22] within the general framework of [6] have found an effective $P = -\frac{1}{3}\rho$ fluid[6]. As in our example above, this result is inconsistent with the tracelessness of $t_{ab}^{(0)}$ (theorem 1 of Sec. II). We believe that the violation of the tracelessness of $t_{ab}^{(0)}$ in [7, 22] is of a similar origin: As discussed in the Introduction, the full content of Einstein's equation has not been utilized, and the requirement that matter satisfy the weak energy condition has not been imposed.

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[1] G. F. R. Ellis, in *General Relativity and Gravitation*, edited by B. Bertotti, F. de Felice, and A. Pascolini (D. Reidel Publishing Company, 1984) pp. 215–288.

[2] A. Ishibashi and R. M. Wald, Class. Quant. Grav. **23**, 235 (2006), arXiv:gr-qc/0509108.

[5] Note also that $t_{ab}^{(0)}$ and $T_{ab}^{(0)}$ are not separately conserved with respect to the background derivative operator, although, of course, by the Bianchi identity for $G_{ab}(g^{(0)})$, we have $\nabla^b(T_{ab}^{(0)} + t_{ab}^{(0)}) = 0$.

[6] Due to the nature of the constructions, these analyses are guaranteed to find an effective stress-energy tensor of FLRW form, with components $t_{\mu\nu}^{(0)}$ which are constant. When one further imposes that this effective stress-energy tensor be conserved, a $P = -\frac{1}{3}\rho$ fluid is automatic.

[3] S. R. Green and R. M. Wald, Phys. Rev. **D83**, 084020 (2011), arXiv:1011.4920 [gr-qc].

[4] T. Buchert, Gen. Rel. Grav. **32**, 105 (2000), arXiv:gr-qc/9906015.

[5] T. Buchert, Gen. Rel. Grav. **33**, 1381 (2001), arXiv:gr-qc/0102049.

[6] R. M. Zalaletdinov, Bull. Astron. Soc. India **25**, 401 (1997), arXiv:gr-qc/9703016.

[7] A. A. Coley, N. Pelavas, and R. M. Zalaletdinov, Phys. Rev. Lett. **95**, 151102 (2005), arXiv:gr-qc/0504115.

[8] T. Buchert and N. Obadia, Class. Quant. Grav. **28**, 162002 (2011), arXiv:1010.4512 [gr-qc].

[9] T. Buchert, J. Larena, and J.-M. Alimi, Class. Quant. Grav. **23**, 6379 (2006), arXiv:gr-qc/0606020 [gr-qc].

[10] S. Rasanen, Phys. Rev. **D85**, 083528 (2012), arXiv:1107.1176 [astro-ph.CO].

[11] C. Clarkson, G. Ellis, J. Larena, and O. Umeh, Rept. Prog. Phys. **74**, 112901 (2011), arXiv:1109.2314 [astro-ph.CO].

[12] B. K. Berger, Annals Phys. **83**, 458 (1974).

[13] J. L. Synge, "Relativity: The general theory," (North-Holland Publishing Company, Amsterdam, 1960) pp. 189–190.

[14] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, IL, 1984).

[15] G. A. Burnett, J. Math. Phys. **30**, 90 (1989).

[16] R. A. Isaacson, Phys. Rev. **166**, 1263 (1968); Phys. Rev. **166**, 1272 (1968).

[17] S. R. Green and R. M. Wald, Phys. Rev. **D85**, 063512 (2012), arXiv:1111.2997 [gr-qc].

[18] D. E. Holz and R. M. Wald, Phys. Rev. **D58**, 063501 (1998), arXiv:astro-ph/9708036.

[19] R. Gowdy, Phys. Rev. Lett. **27**, 826 (1971); R. H. Gowdy, Annals Phys. **83**, 203 (1974).

[20] H. Ringström, Math. Proc. Cambridge Phil. Soc. **136**, 485 (2004), arXiv:gr-qc/0204044 [gr-qc].

[21] C. Charach and S. Malin, Phys. Rev. **D19**, 1058 (1979).

[22] A. Coley and N. Pelavas, Phys. Rev. **D75**, 043506 (2007), arXiv:gr-qc/0607079 [gr-qc].